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CONVEX INTERPOLATION OF CONVEX DATA

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This report contains the mathematical basis of an interpolation technique that constructs a smooth convex interpolant, called an H-Spline, for convex data on the real line. It is shown that an H-Spline always exists and is unique.

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I. INTRODUCTION

In various interpolation problems of experimental and pointwise computer-constructed data the convexity and smoothness (continuous differentiability to the second order) of the interpolant is either implied by the nature of the underlying process or expressly desired. It is well known that classical (Lagrange, Hermite, etc.) and ordinary spline interpolation procedures do not in general produce convex interpolants for convex data and more often than not introduce spurious oscillations between data points. We introduce here a technique for interpolation and display of convex data without such drawbacks. Its basis is provided by an explicit solution of the convex two-point Hermite interpolation problem by a special rational function. To meet the smoothness requirement, a piecewise rational function, called an H-Spline, is constructed. The feasibility of the construction is equivalent to the existence and uniqueness of a solution of a nonlinear system of equations which may be brought into the fixed point form $x = Fx$. Here x is an n-dimensional vector whose dimension depends on the number of data points and whose components represent linear functions of the slopes to be assigned at the internal data points. Section II contains the basic definitions and preparatory remarks. A proof for the existence and uniqueness of an H-Spline for any finite number of data points, based on Brouwer's Fixed Point Theorem and the antitonicity of the operator F, is given in Section III. The general mathematical background for our considerations is covered in[1,2,3]. A companion report[4] describes in detail the actual construction of H-Splines on a computer.

II. DEFINITIONS

We begin with the following definitions:

Definition. A table T: (a_i, y_i), $i = 0, 1, \ldots, n + 1$, such that $a = a_0 < a_1 < \ldots < a_n < a_{n+1} = b$ and $s_0 < s_1 < \ldots < s_{n-1} < s_n$, where $s_i = (y_{i+1} - y_i)/(a_{i+1} - a_i)$, $i = 0, 1, \ldots, n$, is called a convex table.|

Definition. Let R_2^1 be the class of rational functions of order 2 with at most one finite pole. Given a strictly monotone increasing sequence of real numbers $a_0, a_1, \ldots, a_{n+1}$, an H-Spline with joints a_i, $i = o, 1, \ldots, n + 1$, is a function $H(x)$ defined for $a = a_0 \leq x \leq a_{n+1} = b$ that satisfies the following three conditions:

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[1] Collatz, L., Functional Analysis and Numerical Mathematics, (pp. 350-361), Academic Press, New York and London, 1966.

[2] Ortega, J.M. and Rheinboldt, W.C., Iterative Solution of Nonlinear Equations in Several Variables, (pp. 432-446), Academic Press, 1970.

[3] Rall, L. B., Nonlinear Functional Analysis and Applications, (pp. 16-18), Academic Press, 1971.

[4] Egerland, W. O. and Wisniewski, H. L., "Convex Interpolation with H-Splines," to appear.

a. In each interval (a_i, a_{i+1}) for $i = o, 1, \ldots, n$, $H(x)$ is given by some $h_i \varepsilon R_2^1$.

b. $H(x) \varepsilon C^2[a,b]$, i.e., $H(x)$ is twice continuously differentiable on $[a,b]$.

c. $H''(x)>o$, $a\leq x\leq b$, i.e., $H(x)$ is strictly convex on $[a,b]$.|

Since $h_i(x) = (A_{oi}+A_{1i}x + A_{2i}x^2)/(B_{oi} + B_{1i}x)$, $a_i\leq x\leq a_{i+1}$, $A_{2i}\neq o$, $H(x)$, as a plane curve, is represented by an arc of an hyperbola (or a parabola in case $B_{1i} = o$) between adjacent joints. Hence the name H-Spline was suggested.

Definition. Let ℓ and r be positive numbers. A function $f(x)$ solves the "convex two-point Hermite interpolation problem" on the interval $[a,b]$ if (1) $f(a) = o$, $f(b) = o$, (2) $f'(a) = -\ell$, $f'(b) = r$, and (3) $f''(x)>o$, $a\leq x\leq b$.|

According to this definition, the function

$$B(x) = B(x;a,b,\ell,r) = \frac{\ell r(x-a)(x-b)}{\ell(x-a) + r(b-x)}$$

solves the convex two-point Hermite interpolation problem for arbitrary a, b, $a<b$, and positive ℓ and r. $B(x)$ is the simplest H-Spline. We note that

$$B''(x) = \frac{2\ell^2 r^2 (b-a)^2}{[\ell(x-a) + r(b-x)]^3} \tag{1}$$

III. EXISTENCE AND UNIQUENESS PROOF

With the definitions given in Section II, the following theorem holds:

<u>Theorem</u>. Given a convex table T: (a_i, y_i), $i = 0, 1, \ldots, n + 1$, and endconditions y'_o, y'_{n+1}, $y'_o<s_o$, $s_n<y'_{n+1}$, there exists a unique H-Spline such that $H(a_i) = y_i$, $i = o, 1, \ldots, n + 1$, $H'(a_o) = y'_o$, and $H'(a_{n+1}) = y'_{n+1}$.|

<u>Proof</u>. Let y'_i, $i = 1, 2, \ldots, n$, be a sequence of slopes such that $y'_o<s_o<y'_1<s_1\ldots<s_{n-1}<y'_n<s_n<y'_{n+1}$ and consider the function

$$\tilde{H}(x) = L_i(x) + B_i(x),\ a_i \le x \le a_{i+1},\ i = o, 1, \ldots, n, \qquad (2)$$

where $L_i(x) = y_i + s_i(x-a_i)$ and $B_i(x) = B(x;\ a_i, a_{i+1},\ s_i - y_i',\ y_{i+1}' - s_i)$. $\tilde{H}(x)$ is strictly convex on $[a,b]$, $\tilde{H}(a_i) = y_i$, $\tilde{H}'(a_i) = y_i'$, and $\tilde{H}(x) \epsilon C^1[a,b]$. $\tilde{H}(x)$ is an H-Spline if and only if the continuity requirements

$$B''_{i-1}(a_i) = B''_i(a_i) > o,\ i = 1, 2, \ldots, n, \qquad (3)$$

are satisfied. Using (1) and setting

$$y_i' = x_i s_{i-1} + (1-x_i)s_i,\ i = 1, 2, \ldots n,$$

$$K_1 = \left(\frac{a_1 - a_o}{a_2 - a_1}\right)^{1/2} \left(\frac{s_o - y_o'}{s_2 - s_1}\right)^{1/2}$$

$$K_i = \left(\frac{a_i - a_{i-1}}{a_{i+1} - a_i}\right)^{1/2} \left(\frac{s_{i-1} - s_{i-2}}{s_{i+1} - s_i}\right)^{1/2},\ i = 2, \ldots, n - 1,$$

$$K_n = \left(\frac{a_n - a_{n-1}}{a_{n+1} - a_n}\right)^{1/2} \left(\frac{s_{n-1} - s_{n-2}}{y_{n+1}' - s_n}\right)^{1/2},$$

(3) is equivalent to the existence of a solution of the system

$$x_1 = \frac{(1 - x_2)^{1/2}}{(1 - x_2)^{1/2} + K_1} = f_1(x_2)$$

$$x_i = \frac{(1 - x_{i+1})^{1/2}}{(1 - x_{i+1})^{1/2} + K_i x_{i-1}{}^{1/2}} = f_i(x_{i-1}, x_{i+1}),\ i=2,\ldots, n-1$$

$$x_n = \frac{1}{1 + K_n x_{n-1}{}^{1/2}} = f_n(x_{n-1}) \qquad (4)$$

in the open cube I_n: $o<x_i<1$, $i = 1, 2, \ldots n$. This, in turn, is equivalent to the existence of a fixed point in I_n of the mapping $F: I_n \subset R^n \to R^n$ where for $x^T = (x_1, x_2, \ldots, x_n)$ Fx is represented by the column vector

$$Fx = \begin{pmatrix} f_1(x) \\ \vdots \\ f_i(x) \\ \vdots \\ f_n(x) \end{pmatrix}.$$

The mapping F has the following properties:

(P_1) F is continuous on I_n.

(P_2) $FI_n \subset I_n$.

(P_3) F is antitone on I_n.

(P_4) If v, $w \varepsilon I_n$ and $v \leq w$, then v = Fw and w = Fv imply v = w.

(P_1) and (P_2) are obvious, and (P_3) follows from the mean value theorem and $\partial f_i(x)/\partial x_j \leq o$, i, j = 1, ..., n, $x \varepsilon I_n$. To prove (P_4), we observe that the constants K_i, i = 1, ..., n, can be expressed by the components of v and w as follows:

$$K_1 = \frac{1 - v_1}{v_1}(1 - w_2)^{1/2} = \frac{1 - w_1}{w_1}(1 - v_2)^{1/2}$$

$$K_i = \frac{1 - v_i}{v_i}\left(\frac{1 - w_{i+1}}{w_{i-1}}\right)^{1/2} = \frac{1 - w_i}{w_i}\left(\frac{1 - v_{i+1}}{v_{i-1}}\right)^{1/2}, \qquad (6)$$

$$i = 2, \ldots, n - 1,$$

$$K_n = \frac{1 - v_n}{v_n} w_{n-1}^{-1/2} = \frac{1 - w_n}{w_n} v_{n-1}^{-1/2} .$$

The equality of the product of the left sides with that of the right sides in (6) yields, after cancellation,

$$\left(\frac{1 - v_1}{1 - w_1}\right)^2 \prod_{i=2}^{n} \frac{1 - v_i}{1 - w_i} = \left(\frac{v_n}{w_n}\right)^2 \prod_{i=1}^{n-1} \frac{v_i}{w_i}, \qquad (7)$$

a contradiction unless $v_i = w_i$, i = 1, ..., n, i.e., v = w.

We show next that F has exactly one fixed point in I_n. First, let $\delta_o > o$ be such that $\delta_o < \delta_1 = \min\,(m^2(1+m)^{-2};\ (1+M)^{-2};\ 1/2)$, where $m = \min\,(K_1, \ldots, K_n)$ and $M = \max\,(K_1, \ldots, K_n)$. If $x \in I_n(\delta_o)$: $\delta_o \leq x_i \leq 1 - \delta_o$, $i = 1, \ldots, n$, then it is easy to verify that the components of F satisfy the inequalities

$$\delta_o < f_i(x) < 1 - \delta_o,\ i = 1, \ldots, n. \tag{8}$$

$I_n(\delta_o)$ is compact and convex, F is continuous on $I_n(\delta_o)$, and, by (8), $FI_n(\delta_o) \subset I_n(\delta_o)$. Therefore, we may apply Brouwer's fixed point theorem to conclude that F has at least one fixed point x^* in $I_n(\delta_o)$. Furthermore, by the antitonicity of F, the iterations

$$\begin{aligned} v^{k+1} &= Fw^k \\ w^{k+1} &= Fv^k \end{aligned} \qquad k = 0, 1\ldots \tag{9}$$

with initial points $(v^o)^T = (\delta_o, \ldots, \delta_o)$, $(w_o)^T = (1-\delta_o, \ldots, 1-\delta_o)$ define iterates such that ([1], [3], [4])

$$v^o < v^1 \leq \ldots \leq v^{k+1} \leq w^{k+1} \leq \ldots \leq w^1 < w^o. \tag{10}$$

The limits $\lim_{k \to \infty} v^{k+1} = v$ and $\lim_{k \to \infty} w^{k+1} = w$ exist, $v \leq w$, x^* is contained in the order interval $\langle v, w \rangle$, and, by the continuity of F, we have $v = Fw$ and $w = Fv$ from (9). Hence, in view of (5) - (P_4), $v = w = x^*$ is the only fixed point of F in $I_n(\delta_o)$. Since the argument can be repeated with an arbitrary positive $\delta < \delta_o$ and $I_n(\delta_o) \subset I_n(\delta)$, it follows that F has no other fixed point in I_n besides x^*. The unique H-Spline correspondence to $(x^*)^T = (x_1^*, x_2^*, \ldots, x_n^*)$ is given by $H(x) = \tilde{H}(x)$ in (2) with $y_i' = x_i^* s_{i-1} + (1 - x_i^*)s_i$, $i = 1, \ldots n$. This completes the proof of the theorem. |

<u>NOTE</u>:

An outline for the proof of the existence and uniqueness of an H-Spline for a given convex table was presented at the Twentieth Conference of Army Mathematicians at the U.S. Army Natick Laboratories, Natick, Massachusetts, May 1974.

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1. Collatz, L., Functional Analysis and Numerical Mathematics, (pp. 350-361), Academic Press, New York and London, 1966.

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